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Elastic modulus

Part-2 : Experimental measurement and applications

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Outline

- Elasticity vs resonance – short video (You tube)
- Methods of measurement of Young's modulus
- Flexural vibrations of a bar - theoretical aspects
- Experimental determination of 'Y' for a steel bar.
- Other Moduli of elasticity
- Stiffness and engineering applications

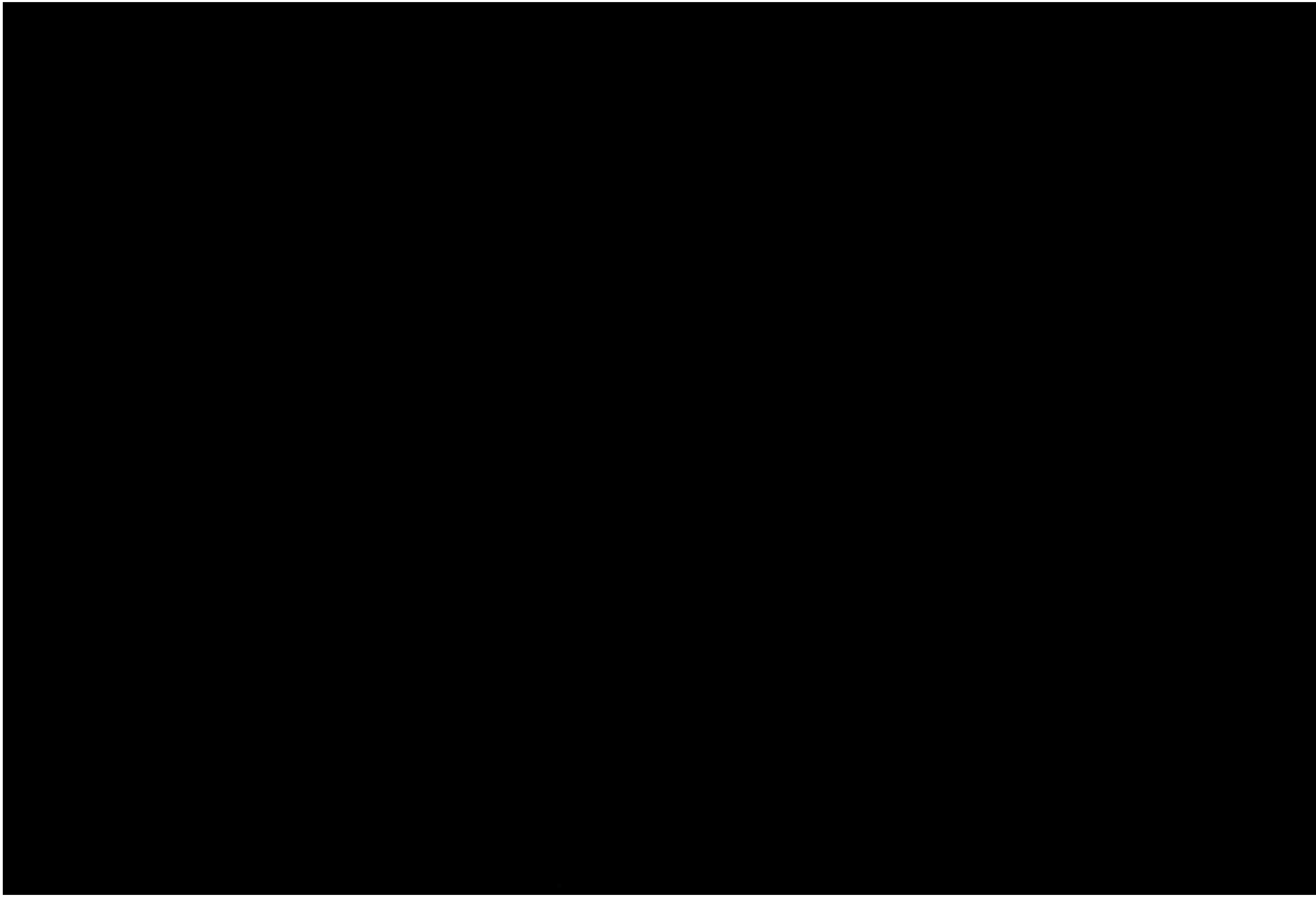
Elasticity vs resonance and structure collapse

Tacoma Narrows Bridge

From Tacoma to Gig harbor, Washington, USA

Opened for traffic : July 1, 1940
Collapsed on: Nov.7,1940

Source : You tube



Source : You tube

The two new bridges in 2007

Source : Wikipedia

Importance of measurement

“When you **can measure** what you are speaking about, and **express** it in numbers, you know **something** about it,
when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind ………..“

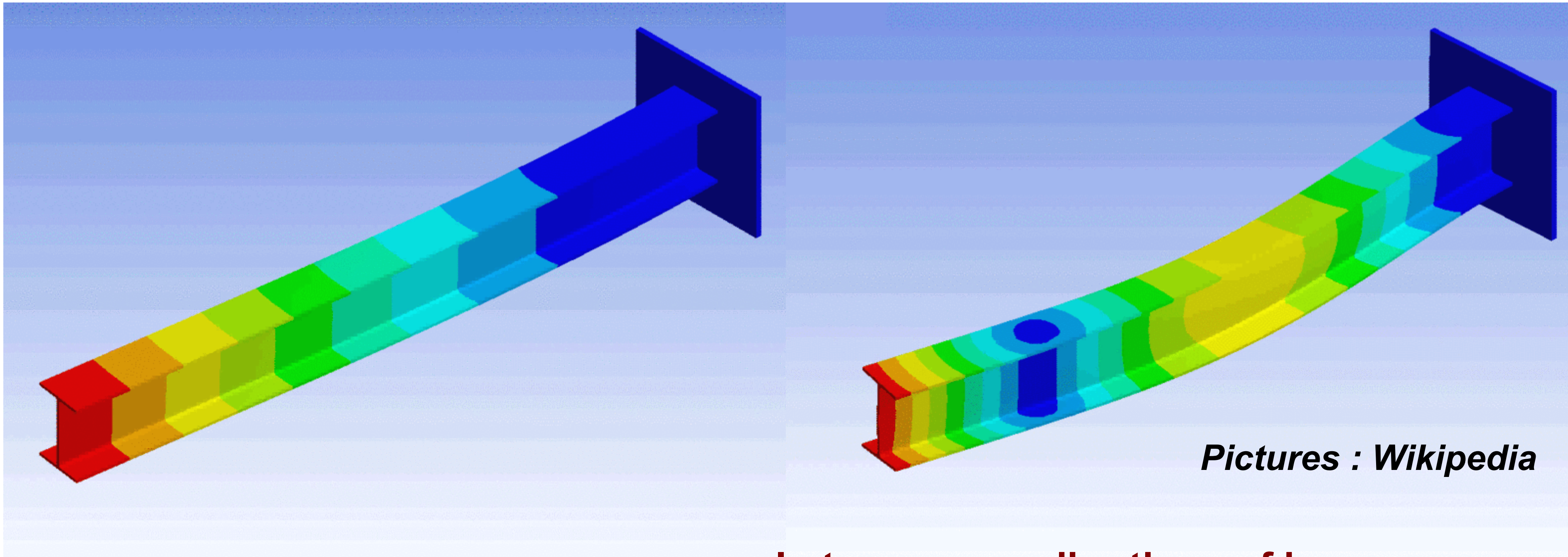
Sir William Thomson, (Lord Kelvin) 1883

Experimental determination of Young's modulus

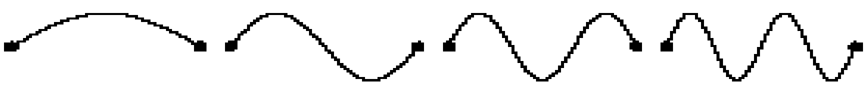
No	Technique
1	Stress-strain curve – tensile testing instrument
2	Direct measurement of Stress - strain in long wires-Searle's method
3	Cantilever method – (i) depression by pin and microscope, (ii) deflection by optical measurement, (iii) free oscillations
4	Measurement of bending moment of supported bar – Koenig's method
5	Measurement of velocity of sound- piezoelectric method
6	Tip induced deformation experiments
7	Flexural vibrations of a bar

Ref: (i) A text book of practical physics by MN Srinivasan, S Balasubramanian and R Ranganathan,S Chand & sons, New Delhi, 2003
(ii) Determination of elastic moduli of sintered metal powder compacts using an ultrasonic method. P Ramarao and AA Krishnan, J. Scientific & industrial Research, 1959, Vol 18B(6), pp 260-261

Flexural Vibrations of a bar, supported at one end



Pictures : Wikipedia



Transverse vibrations in a stretched string

In transverse vibrations of bars, "Elasticity" is responsible for restoring successive portions of the bar to their original position, where as for transverse vibrations of strings, "Tension" is responsible.

The theory of vibrations of bars, even when simplified to the utmost by the omission of unimportant quantities, is decidedly more complicated than that of perfectly flexible strings.

Lord Rayleigh, The theory of sound (1894)

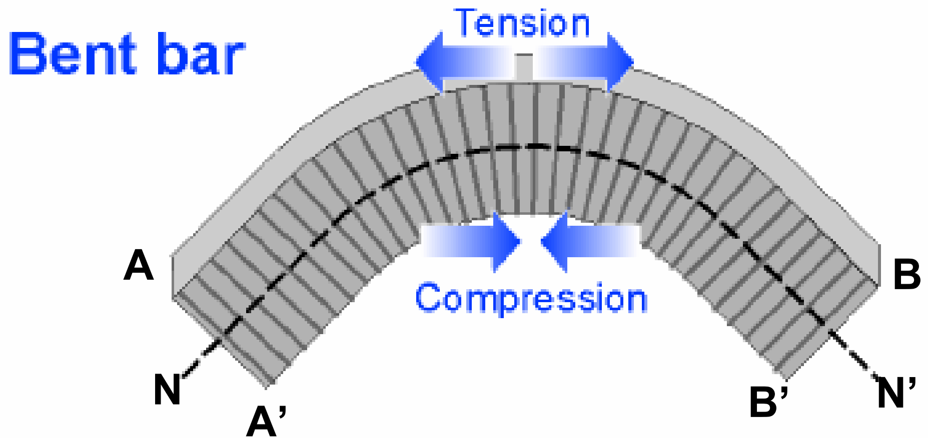
Mathematical steps to calculate the Young's modulus by flexural vibrations of a bar

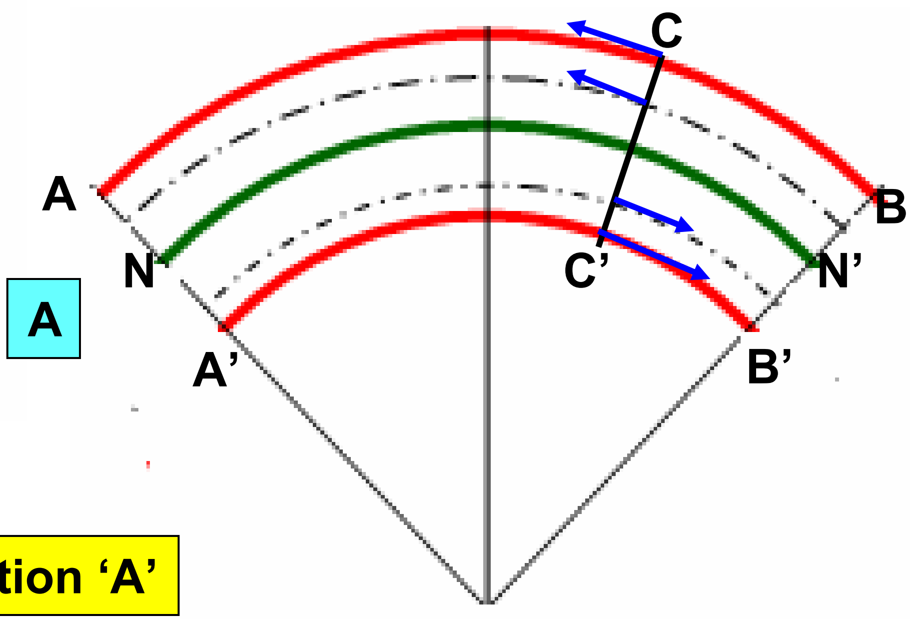
Step No.	Operation
01	Equations for bending moment and shear forces
02	Wave equation and the general solution
03	Application of boundary conditions at clamped and free ends and equation for fundamental frequency.

Sources :

- *"A simple experiment on flexural vibrations and Young's modulus measurement" by Salvatore Ganci, Physics Education 44(3), pp 236-240, 2009*
- *Unified Physics, SL Gupta and Sanjeev Gupta, Vol.1, Mechanics, Waves and Oscillations (1996)*
- *Kit developed for doing experiments in physics- Instruction manual by R Srinivasan and KRS Priolkar- March 2010 (Sponsored by Indian Academy of Sciences, Bangalore, Indian National Science Academy, Delhi and The National Academy of Sciences India, Allahabad)*

Origin of bending moment in a bar





Consider a bar of rectangular cross section 'A'

When the bar is bent, length of neutral line (NN') is not changed. Tensile and compressive forces increase as we move away from neutral surface.

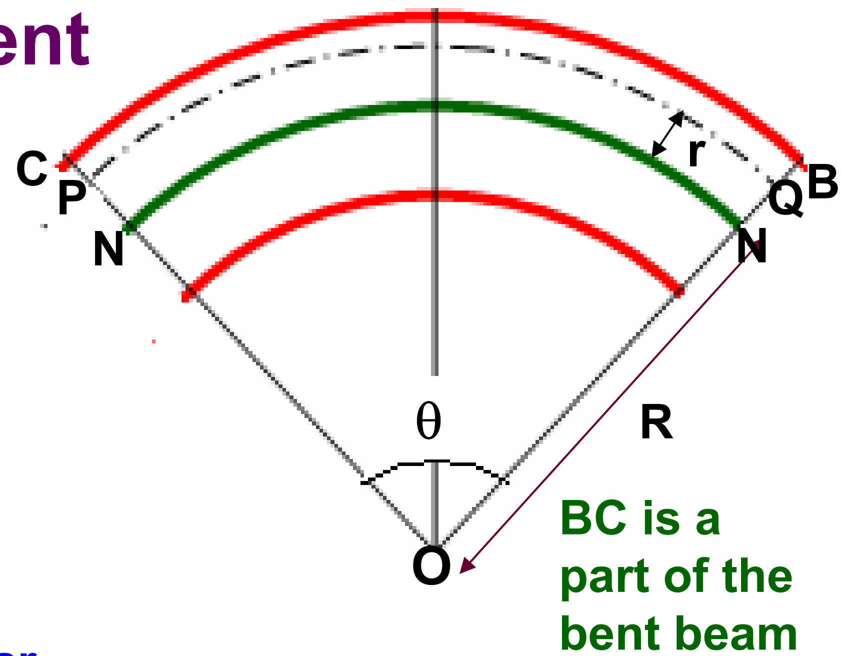
The longitudinal strains in the layers increase from zero (neutral layer) to a maximum at the upper and lower surfaces and accordingly the stresses.

The forces of compression below the neutral surface and elongation above the neutral surface constitute a couple and the magnitude of the moment of the couple is known as "bending moment"

At equilibrium, the anticlockwise internal couple is balanced by the clockwise external couple

Equation for bending moment

<u>Method</u> : Strain - stress - force - torque evaluation at elementary level and integration



The bending moment of the bar is given by

$$\tau = Y A k^2 (\partial^2 y / \partial x^2)$$

'y' is the transverse displacement of the bar at position 'x'

$A k^2 = \int r^2 dA$ (geometric moment of inertia)

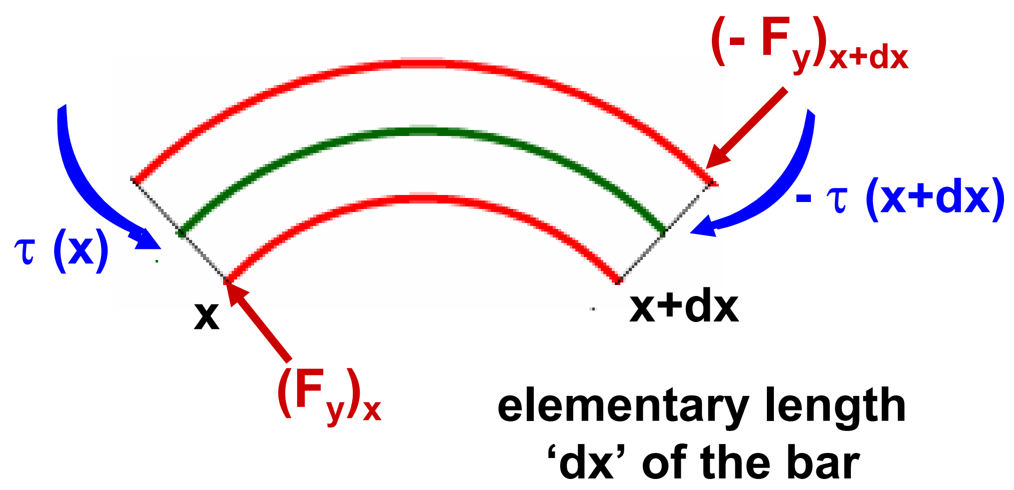
NN : Neutral surface
R : Radius of curvature of neutral surface from "O" the centre of curvature.
θ : Angle subtended by neutral surface at the centre "O"

Bending torque is second partial derivative of position (x) and hence is not the same at every part of the bar.

Equation for shear forces

Distortion of the bar produces bending moments as well as shear forces.

Method : Define torques and forces at the elementary level and apply the condition



Condition:

When the bar is vibrating, it is in dynamic equilibrium and the torque and the shear forces must be such as to produce no net turning moment.

$(F_y)_x$	= upward shear force at left end (+)
τ_x	= anti clockwise torque at left end (+)
$(F_y)_{x+dx}$	= downward shear force at right end (-)
τ_{x+dx}	= clockwise torque at right end (-)

$$F_y = (\partial \tau_x / \partial x)$$

Shear forces are partial derivative of the bending torque

Substituting the value of τ_x ,

$$F_y = Y\,Ak^2\,(\partial^3 y / \partial x^3)$$

Shear forces are third partial derivative displacement with respect to position

Wave equation

Method : Differentiate the force equation and apply Newton's second law (F=ma)

i.e. $dF_y = \rho A\, dx\, (\partial^2 y / \partial t^2) = \{Y A k^2 (\partial^4 y / \partial x^4)\}\, dx$

Wave equation

$$(\partial^2 y / \partial t^2) = (Y/\rho)\, k^2 (\partial^4 y / \partial x^4) \quad (1)$$

$$(\partial^2 y / \partial t^2) = v^2 k^2 (\partial^4 y / \partial x^4) \quad (2)$$

ρ= density;
A = area of cross section
Y = Young's modulus
Ak^2 = geometric moment of inertia
$v = \sqrt{(Y/\rho)}$ = Velocity

Assuming harmonic vibration with frequency, 'ω', the wave equation can be re-written as

$$Y (A k^2) (\partial^4 y / \partial x^4) - \rho A \omega^2 y = 0 \quad (3)$$

Transverse vibrations of a rigid bar contains fourth partial derivative.

For comparison :

Wave equation for transverse vibrations of strings :

$$(\partial^2 y / \partial t^2) = v^2 (\partial^2 y / \partial x^2)$$ → Second derivative

The solution is $y = \{ f(vt + x)\}$.

Solution to the wave equation

Method : general solution – application of boundary conditions to get constants- frequency equation

For a rectangular bar, $Ak^2 = bd^3/12$ where 'b' is width and 'd' is thickness.

Upon simplification, the eqn. (3) becomes

$$(d^4y / dx^4) - \alpha^4 y = 0$$

where $\alpha^4 = 12\, \rho\omega^2\, Yd^2$

The general solution of this eqn. is

$$Y(x) = A \cosh(\alpha x) + B \sinh(\alpha x) + C \cos(\alpha x) + D \sin(\alpha x)$$

Boundary conditions

@ <u>Clamped end:</u>
- There can be no displacement and slope at all time.
- $y = 0$ and $\partial y / \partial x = 0$

@ <u>Free end:</u>
- There can be neither external torque nor a shearing force.
- $\partial^2 y / \partial x^2 = 0$ and $\partial^3 y / \partial x^3 = 0$

The four constants A,B,C and D are determined from boundary conditions.

1. At the clamped end: $x = 0$ (i) $y = 0$ and (ii) $dy / dx = 0$
2. At the free end : $x = l$ (iii) $d^2y / dx^2 = 0$ and (iv) $d^3y / dx^3 = 0$

Applying the boundary conditions(1), we get
$A+C = 0$
$B+D = 0$

Applying the boundary conditions(2), we get
$A (\cosh \alpha l + \cos \alpha l) + B (\sinh \alpha l + \sin \alpha l) = 0$
$A (\sinh \alpha l + \sin \alpha l) + B (\cosh \alpha l + \cos \alpha l) = 0$

Eliminating trivial solutions, we get $\alpha_1 l = 1.875$ for the first solution.

The lowest frequency is given by $\omega_1 = (1.875 / l)^2 (Y d^2 / 12 \rho)^{1/2}$

Formula:

$$Y = (4\pi^2 / 1.875^4)(12 \rho / d^2)(f l^2)$$

l^2 and $1/f$ have linear dependency

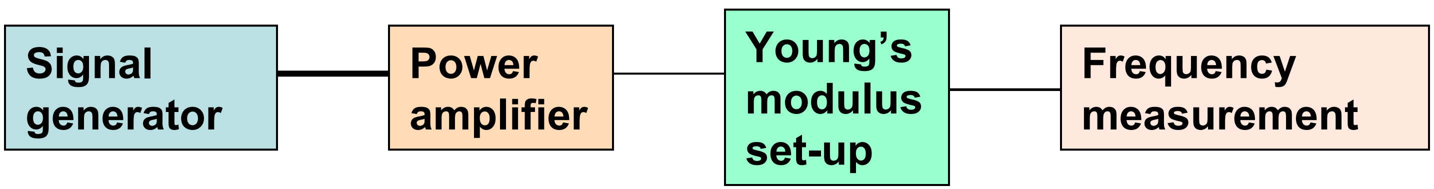
ρ : density
d: thickness
F: lowest frequency
l: length

Experimental set-up and measurement of Young's modulus

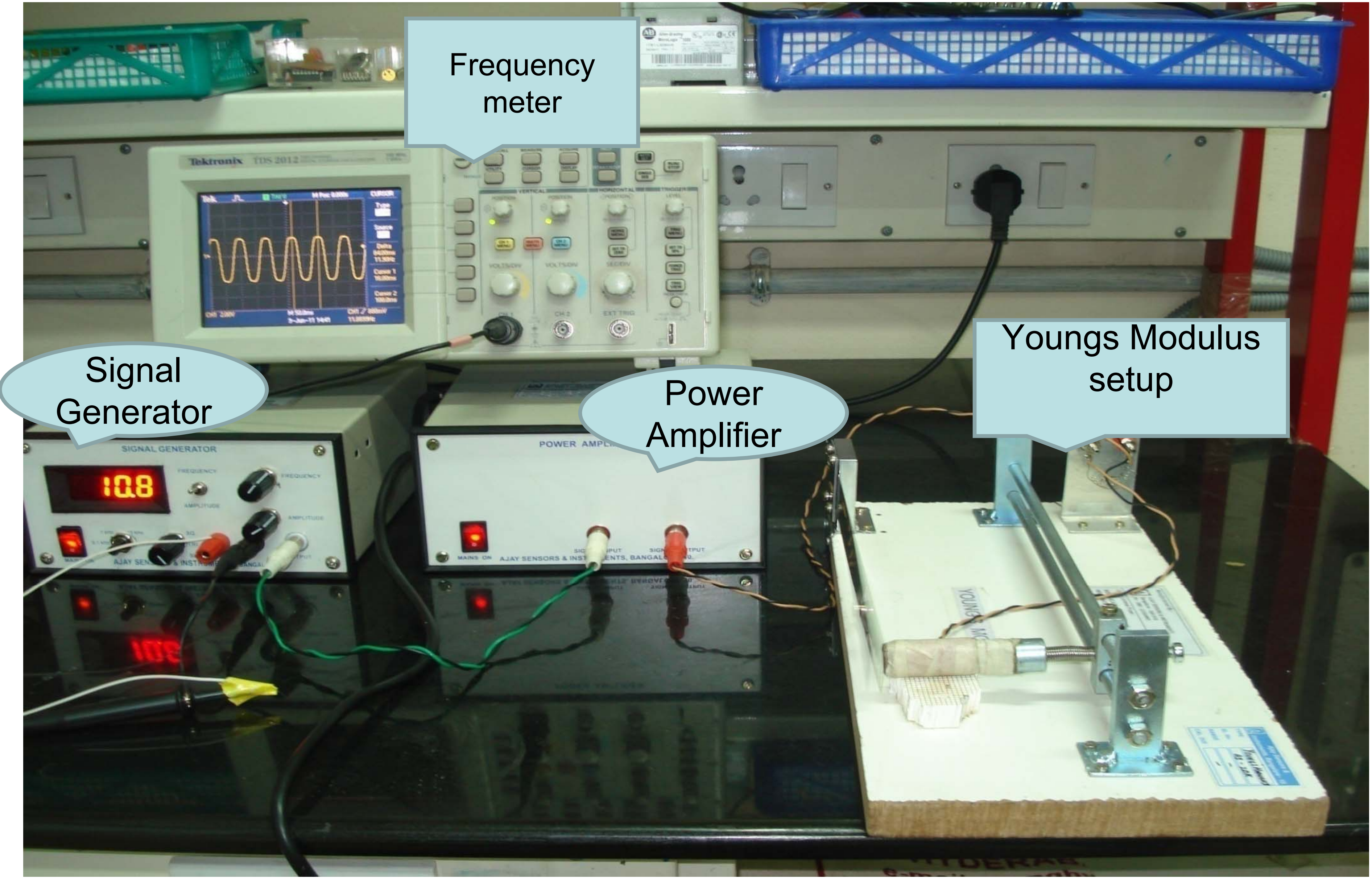
Aims of the Experiment

1. To measure the first frequency of vibration at different lengths and show that l^2 and $1/f$ have linear relationship.
2. To determine the average value of the slope of the curve (fl^2)
3. To determine the Young's modulus of the material of the bar by using the formula

Component details of experimental set-up



Experimental set-up

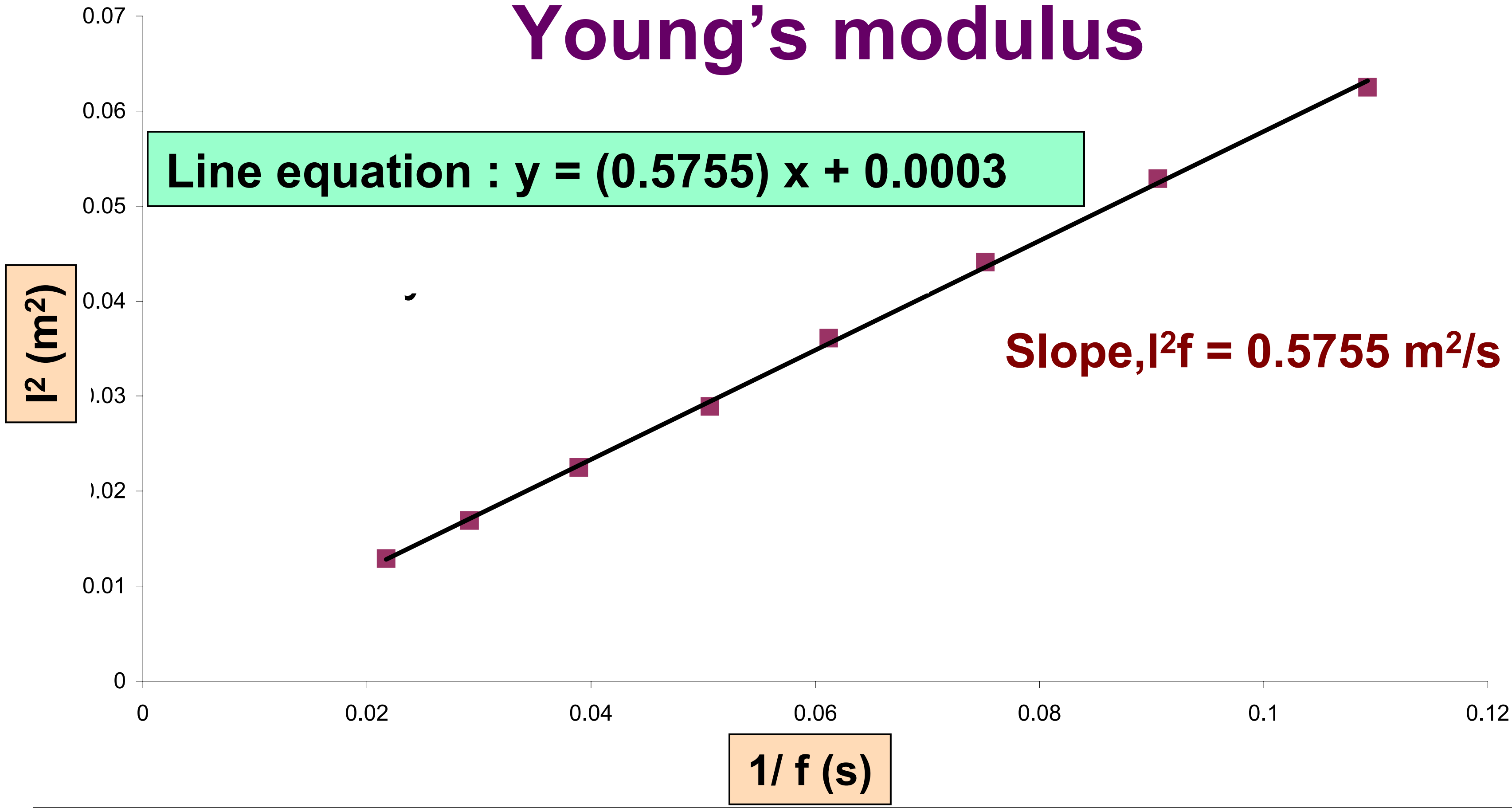


Flexural vibrations of a steel bar

Observations

Sl.No	Length, l (cm)	Frequency, f (Hz)	l^2 (m^2)	1/f (sec)
1	25	9.15	0.0625	0.1093
2	23	11-04	0.0529.	0.0906
3	21	13.3	0.0441	0.0752
4	19	16.34	0.0361	0.0612
5	17	19.76	0.0289	0.0506
6	15	25.7	0.0225	0.0389
7	13	34.5	0.0169	0.0292
8	11	46.03	0.0129	0.0217

Plot of l2 Vs 1/f and determination of Young's modulus



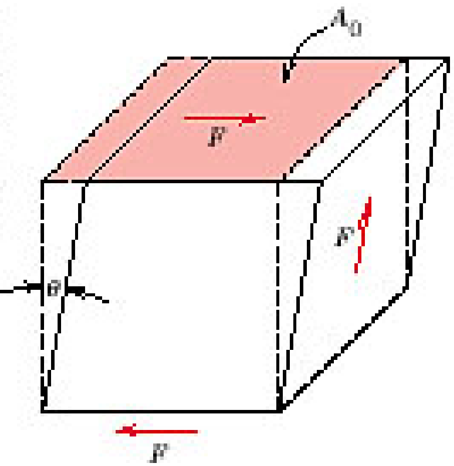
$Y = (4\pi^2 / 1.875^4)\ (12\ \rho / d^2)\ (l^2 f)^2 = 38.33\ (\ \rho / d^2)\ (l^2 f)^2$

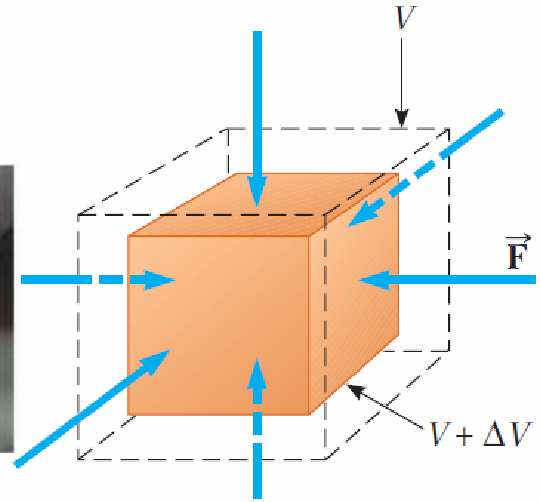
Thickness, (d) = 0.74×10^{-3} m
Density, (ρ) of steel = 7850 kg / m^3
Slope ($l^2 f$) = 0.5755 m^2/s

Substituting the values,
Y = 182 GPa

Other moduli of elasticity

Shear modulus

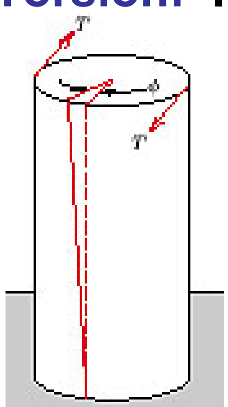




Bulk modulus

The cube undergoes a change in volume; but no change in shape

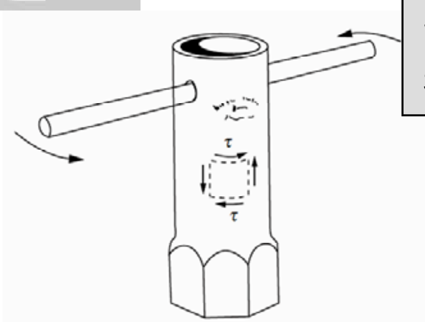
Torsion: like shear.



Shear stress: τ = F / A_o
F: Applied parallel to upper and lower faces each having area A_0.

Shear strain: $\gamma = \tan\theta$
(× 100 %) θ is strain angle

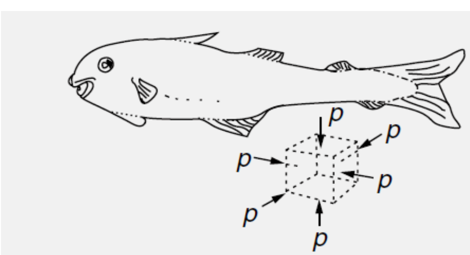
Shear modulus, G = shear stress / shear strain



Pure shear, $\tau = \frac{F_s}{A}$

Bulk stress = applied pressure = p
Bulk strain = change in volume = - (Δv/v)}

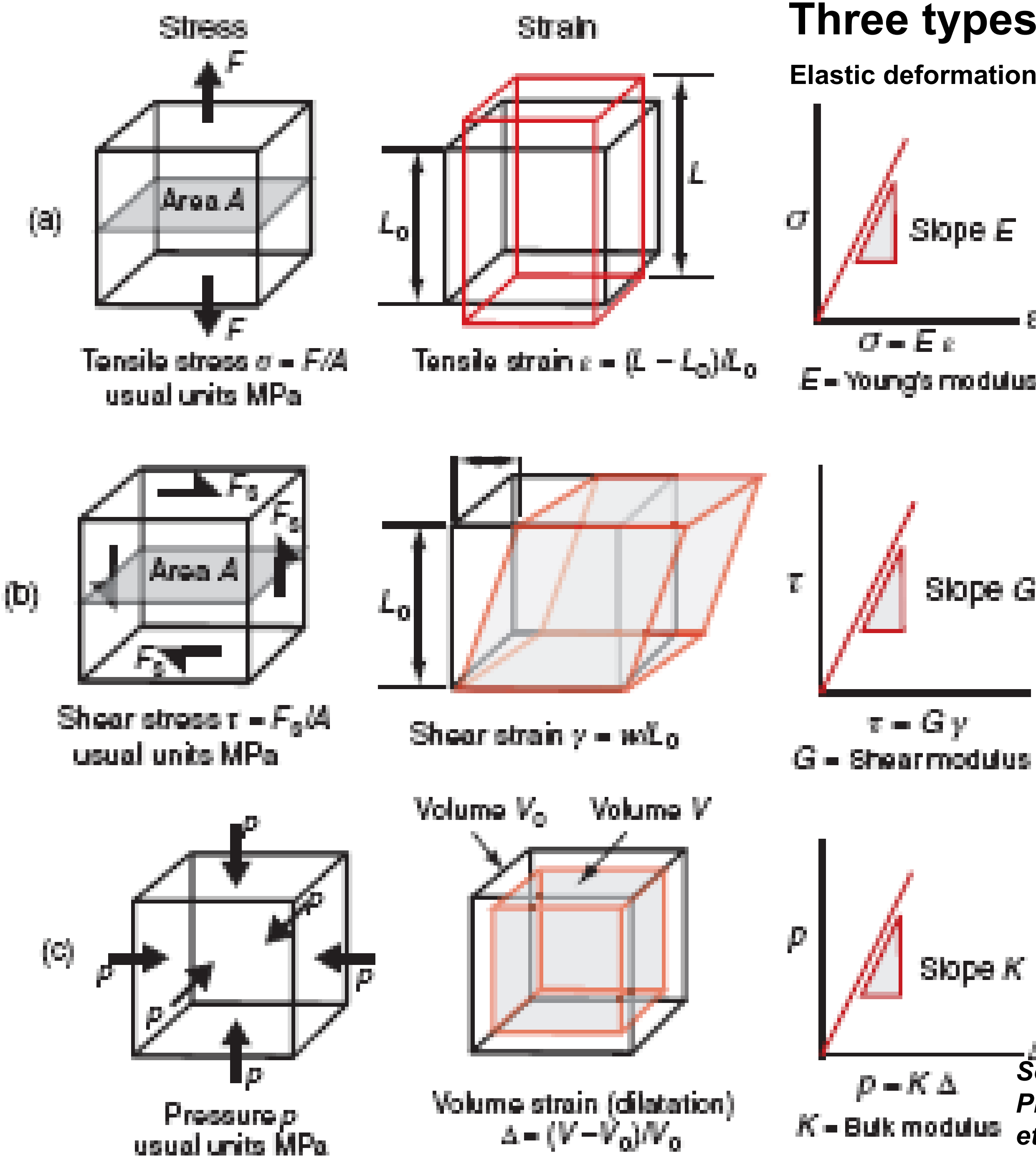
Bulk modulus K = Bulk stress / bulk strain



Hydrostatic pressure, $p = -\frac{F}{A}$

Source: Materials Engg,Science, Processing and Design by Michael Ashby et al
Butterworth-Heinemann, 2007

Three types of elastic moduli



Tensile stress, Tensile strain and Young's modulus

Poisson's ratio

- ❖ **σ = (lateral strain) / (vertical strain)**
- ❖ **Dimensionless**

Shear stress, Shear strain and Shear modulus

Bulk stress, Bulk strain and Bulk modulus

Source: Materials Engg,Science, Processing and Design by Michael Ashby et al Butterworth-Heinemann, 2007

Relationship between Elastic moduli for homogeneous isotropic materials

For homogeneous isotropic materials simple relations exist between elastic constants, that allow calculating them all, as long as two constants are known.

E : Young's modulus;
G : Shear modulus ;
K : Bulk modulus;
σ : Poisson's ratio

Conversion formulas

	(E,K)	(E,G)	(K,G)	(E, σ)	(K, σ)	(G, σ)
E =	--	--	9KG / (3K+G)	--	3K (1-2σ)	2G (1+σ)
K =	--	EG / 3(3G - E)	--	E / 3(1-2σ)	--	2G(1+σ) / 3(1-2σ)
G =	3KE / (9K-E)	--	--	E / 2(1+ σ)	3K (1-2 σ) / 2(1+ σ)	--
σ =	(3K – E) / 6K	(E/2G) - 1	(3K – 2G) / 2(3K+G)	--	--	--

Source: Wikipedia

Engineering applications of elastic properties

Elastic modulus	Significance
Young’s Modulus (E or Y)	Resistance to stretching
Shear Modulus (G)	Resistance to twisting
Bulk Modulus (K)	Resistance to hydrostatic compression

Stiffness

❖ Stiffness (S = dF/dr) represents the rigidity of an object — the extent to which it resists deformation in response to an applied force.

❖The stiffness of an engineering component depends not only on the Young's modulus of the material, but also on how it is loaded (tension, or bending) and the shape and size of the component.

❖ A body may also have a rotational stiffness, (ratio of applied torque to angle of rotation), shear stiffness (ratio of applied shear force to shear deformation) and torsional stiffness (ratio of applied torsion moment to angle of twist)

❖ A *stiff* material has a high Young's modulus (e.g. diamond).
❖ A *flexible* material has a low Young's modulus (e.g. rubbers).

Source: Wikipedia

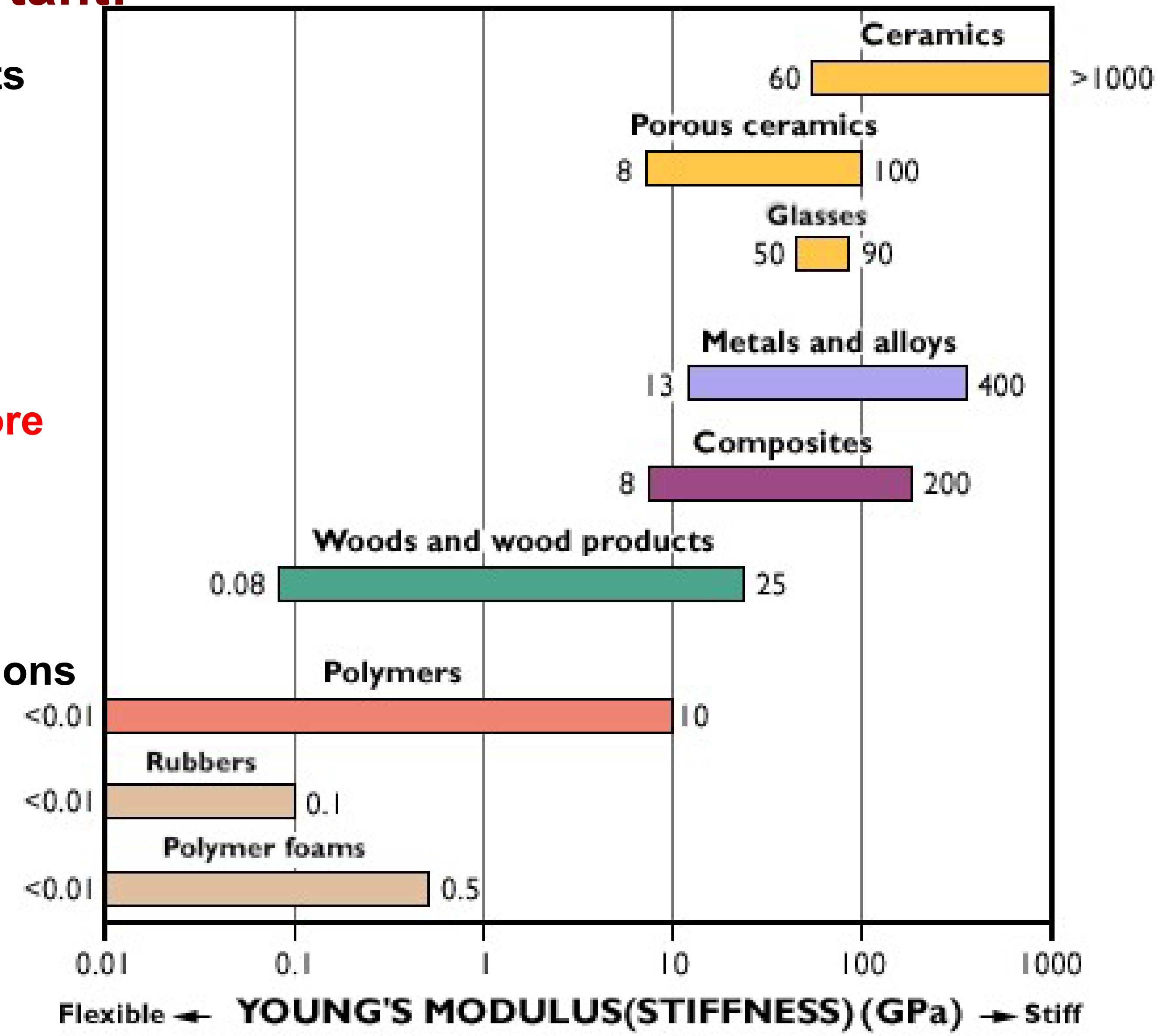
Typical design issues

Stiffness is important:

❖ In designing products which can only be allowed to deflect by a certain amount
(e.g. bridges, bicycles, furniture).

❖ In springs, which store elastic energy
(e.g. vaulting poles, bungee ropes).

❖ In transport applications stiffness is required at minimum weight
(e.g. aircraft, racing bicycles) .



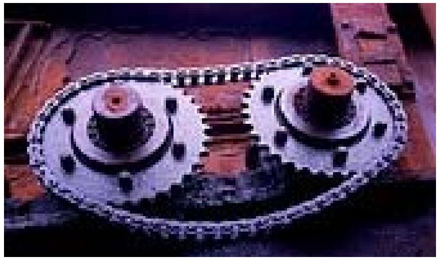
Source: Wikipedia

Engineering Applications – common examples

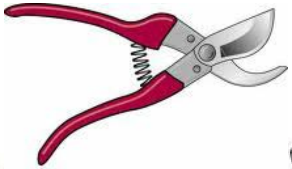
Design criteria: Transfer bending moments, shearing forces and compressions












Pictures from Internet

Thank you